Effect of symmetry breaking on "chaotic" eigenfunctions

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The theory of random matrices predicts that the eigenvector statistics of quantum operators associated with chaotic dynamics should undergo a rapid transition from one universality class to another as a symmetry of the system is gradually broken. We show by a numerical calculation that the transition in strength correlations of the eigenvector components are identicial to the random-matrix predictions for time-reversal violations. This transition turns out to be governed by the same parametrization as in the case of spectral fluctuations of these systems but the speed of transition is different for the two cases.

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I. INTRODUCTION

There has been considerable interest in the spectral and strength correlations of quantum-mechanical operators associated with systems whose classical analogs are fully chaotic. Various studies indicate that, in the semiclassical limit $\hbar \to 0$, correlations of a few eigenvalues and eigenvectors show universal behavior; they depend only on the symmetries existing in the system and are independent of all other details of the distribution of individual matrix elements. This behavior of short- and long-range correlations can be modeled by the universality classes of random matrices, i.e., the matrices whose elements are random variables with given probability laws.

The three important universality classes well suited for the autonomous Hamiltonians are the Gaussian orthogonal ensemble (GOE) of real symmetric matrices, the Gaussian symplectic ensemble (GSE) of quaternion self-dual Hermitian matrices, and Gaussian unitary ensembles (GUE) of general Hermitian matrices. The first two of these are for time-reversal (T) invariant systems and the third for time-reversal noninvariant (TRNI) systems. For the evolution operators of quantum maps there are three analogous circular ensemble models, viz., the COE of symmetric unitary matrices, the CSE of quaternion self-dual unitary matrices, and the CUE of general unitary matrices. Dyson and Mehta [1,2] have calculated the correlation functions of all orders in these ensembles. The GOE works well for nuclear, atomic, and molecular data [3], but there are excellent confirmations of all the universality classes in chaotic systems with few degrees of freedom for autonomous systems [4–6] as well as quantum maps [7,8].

These ensembles, however, may not be appropriate when a symmetry of the system is only partly broken [9]. This is because the spectral and strength fluctuations in a complicated system depend on the good symmetry of the system, changing therefore from one pattern to another as the symmetry parameter is slowly varied. For example, for partial T violation in autonomous systems, ensembles intermediate between GOE and GUE, or between GSE and GUE, may be appropriate [9–11,4]. Similar considerations for approximate quantum numbers (such as isospin in nuclear spectra [12] and LS quantum numbers in atomic spectra [13]) may lead to a transition from a superposition of several independent GOE's to a single GOE. In order to model such cases, Dyson [14] introduced Brownian motion ensembles of random matrices in which the symmetry-breaking parameter τ (say, the ratio of the squared norms) plays the role of "time" and the above universality classes are obtained as stationary ($\tau \to \infty$) limits. This conjecture which is now well confirmed [9–11,15] not only lends credence to universality above but also gives new techniques for detecting small symmetry breakings in real systems. It is shown moreover that, for very small τ and large N (dimension of the matrix), the transition will be smooth as a function of a local parameter Λ which measures the mean-square symmetry admixing matrix element (τv^2) in units of local average spacing D; $\Lambda = \tau v^2/D^2$ ($= \tau N^\alpha$, where $\alpha > 0$). For GOE$\to$GUE and GSE$\to$GUE (similarly for COE$\to$CUE and CSE$\to$CUE) transitions, the correlation functions of all orders have been obtained [10,15]. This transition theory when applied to nuclear data has given sharp bounds on T violation in the nuclear interaction [11]. Moreover, now we also have compelling evidence (analytical as well as numerical) [16,19] that the symmetry breaking clearly manifests itself in spectral correlations in conformity with the random-matrix predictions.

The success of random-matrix theory (RMT) in modeling the eigenvalue spectra, both in exact as well as partially violated symmetry cases, encourages one to hope the same for eigenvector statistics too. In fact, a recent study [20] for kicked tops with exact symmetries shows that eigenvector statistics in this case has a universal nature and behaves in a similar way as that for the exact symmetry classes of RMT. The eigenvector statistics for random-matrix ensembles, for symmetry-preserving cases, has already been worked out [11,17,18]. In the large matrix dimensionality limit, $N \to \infty$, the probability density for one component of an eigenvector for these cases can be expressed by a χ^2_β distribution of degree β with $\beta = 1$ for the GOE and $\beta = 2$ for the GUE. This result is also valid for the circular ensembles in the large-N limit. A similar formulation for the Brownian motion ensembles, i.e., the random-matrix ensembles with a small symmetry-violation, is not yet fully available. There have been many attempts in this direction. The first one was by French *et al.* (later referred to as FKPT) [11], who gave an approximate theory for the probability density of one component of an eigenvector and the variance. Later on, Zyczkowski and Lenz [21] suggested a different form of the one-parameter family of interpolating functions. Their formulation was based on the assumption that the real and imaginary parts of the eigenvector components are distributed independently according to two different Gaussians. But the dependence of the distribution of the eigenvector component on its modulus only (due to gauge freedom) makes this assumption inadequate as recently pointed out by Sommers and Iida (later referred to as SI) [22].

In an attempt to remove this inadequacy, Sommers and Iida [22] proposed a new formulation for the probability density of one component of an eigenvector for transition ensembles, in the large matrix dimensionality limit. Recently, a numerical verification carried out by Zyczkowski [24], for random-matrix ensembles, indicated validity of the formulation. In this paper, we numerically apply this formulation to symmetry-breaking cases of quantum chaotic systems and find its validity here too. Moreover our calculations for a generic quantum chaotic map, namely kicked rotor, show that the eigenvector statistics not only in the exact symmetry cases but also in the transition cases has a universal nature and can be well modeled by random-matrix formulation. We reached a similar conclusion about eigenvalues statistics as well [16].

We choose the kicked rotor system for our numerical study, as it has been an active model of research, containing a variety of features such as localization, resonance, dependence of the spectra on the number theoretical properties, etc., and has been used as a model for a very wide range of physical systems [23]. Besides, the kicked rotor can also display discrete symmetries such as parity and time reversal [7,16]. This will enable us to study the way small violations of these symmetries reveal themselves in the distribution of eigenfunctions. The Hamiltonian associated with kicked rotor dynamics is time periodic, which results in the time-independent nature of the time-evolution operator (if considered for integer multiples of the kicking period). The quantum dynamics of the kicked rotor, therefore, can be studied in terms of eigenvalues and eigenvectors of the time-evolution operator.

It is worth mentioning here that so far random-matrix results are derived only for ensembles of Hermitian matrices and we have dealt with ensembles of unitary matrices (i.e., the time-evolution operator). But a good agreement of our numerical results with those obtained analytically for Gaussian ensembles implies the formulation is valid for circular ensembles too, in the large matrix dimensionality limit. This is not surprising, as we have already seen that at least all the spectral measures for both Gaussian as well as circular ensembles tend to have the same form in the semiclassical limit [15].

To study the variation of statistical properties of eigenvector components due to a weak symmetry breaking, we proceed as follows. First, in Sec. II, we briefly review the dynamics of a kicked rotor, both classical as well as quantum, and the associated symmetries. Section III deals with a brief history of RMT for eigenvector components; here we review the formulations given by both FKPT and SI. The reason for including FKPT formulation is that it describes the smooth transition in strength correlations in terms of the parameter, which is used for the same purpose in spectral correlation too; we studied the spectral correlations in a quantum kicked rotor (QKR) in terms of this parameter. This is followed by a numerical study of the eigenvector statistics of a kicked rotor, given in Sec. IV.

II. THE KICKED ROTOR: CLASSICAL AND QUANTUM DYNAMICS

The kicked rotor can be described as a pendulum subjected to periodic kicks (with period T) with the following Hamiltonian:

$$H=\frac{(p+\gamma)^2}{2}+\frac{K}{4\pi^2}\cos(2\pi\theta+\phi)\sum_{n=-\infty}^{\infty}\delta(t-nT)\ , \qquad (1)$$

where K is the stochasticity parameter. For simplicity we set $T=1$. The parameters γ and ϕ are introduced in the Hamiltonian in order to mimic the effects of the time-reversal (T) and the parity (P) symmetry breaking in a quantum system.

Integration of the associated equations of motion for $t=(n-1/2)$ and $t=(n+1/2)$ and rescaling $p/2\to p$ gives the classical map,

$$\begin{aligned} p_{n+1}&=p_n+\frac{K}{4\pi}\sin\left[2\pi\left[\theta_n+\phi+p_n+\frac{\gamma}{2}\right]\right]\ ,\\ \theta_{n+1}&=\theta_n+p_n+p_{n+1}+\gamma\ , \end{aligned} \qquad (2)$$

The map is area preserving and is invariant under the discrete translation $(\theta\to\theta+1,p\to p+1)$. The other two discrete symmetries in the classical kicked rotor are time reversal $T(p\to -p,\theta\to\theta,t\to -t)$ and parity $P(p\to -p,\theta\to -\theta)$, preserved even for nonzero values of γ and ϕ. It is because the change of $p\to p+\gamma$ or $\theta\to\theta+\phi/2\pi$ is a canonical transformation and therefore does not affect the classical Hamiltonian. But, as explained later, the nonzero values of γ and ϕ may break respective symmetries in the quantum kicked rotor [16]. The classical map depends only on parameter K. Under the variation of K, the dynamics changes from integrable $(K=0)$ to near-integrable $(0<K<4.5)$ to large-scale chaos $(K>4.5)$.

Floquet's theorem enables us to describe the related quantum dynamics by a discrete time evolution operator $U=GBG$, where

$$\begin{aligned} B&=\exp(-iK\cos(2\pi\theta+\phi)/4\pi^2\hbar)\ ,\\ G&=\exp(-i(p+\gamma)^2/4\hbar)\ . \end{aligned} \qquad (3)$$

The nature of the quantum dynamics and therefore the statistical properties of the associated quantum operators depend on $\hbar$ and K. For a rational value of $\hbar/2\pi$, the dynamics can be confined to a torus while for irrational value it takes place on a cylinder [23]. We employ torus boundary conditions ($\theta'=\theta+1, p'=p+M$) by taking $\hbar/2\pi=M/N$; both p and θ then have discrete eigenvalues and U can be reduced to a finite N-dimensional matrix of the form [23]

$$U_{mn}=\frac{1}{N^2}\sum_j \exp\left[-i\frac{K}{4\pi^2\hbar}\cos\left[\frac{2\pi j}{N}+\phi\right]\right] \times \sum_{l,l'=-N_1}^{N_1} \exp(-i[\pi^2\hbar(l^2+l'^2)-\pi\gamma(l+l')])\exp\left[-i\left[\frac{l(m-j)+l'(j-n)}{N}\right]\right], \tag{4}$$

where $N_1=(N-1)/2$ (with N odd) and $m,n=-N_1,-N_1+1,\ldots,N_1$.

In contrast to the classical dynamics, the quantum dynamics, if restricted to a torus, can be affected by the transformation $p\to p+\gamma, \theta\to\theta+\phi/2\pi$. The reason is that the quantum Hamiltonian, acting in a finite Hilbert space, is no longer invariant under a unitary transformation [i.e., $T^\dagger UT\neq U$ or $P^\dagger UP\neq U$ for the values of $j\pi/N<\phi<(j+1)\pi/N, l\hbar<\gamma<(l+1)\hbar$ with both j and l as integer]. Our numerical analysis further indicates that invariance of the quantum dynamics under a symmetry breaking depends not only on respective values of γ and ϕ but also on K. More precisely, it is the relative values of three parameters, namely, K, $\hbar$, and N, which can affect the quantum dynamics quite significantly. The earlier studies [23] have shown that for $K^2>N\hbar$ ($=2\pi M$) (the strong chaos limit) the eigenstates are fully extended in momentum space. We also know that the spectrum as well as the distribution of eigenvector components in this case, with parameters γ and ϕ chosen to preserve either exact or partially violated symmetry, can be modeled by the random-matrix theories (RMT) [7,16]. In the opposite limit of weak chaos, namely, $K^2\ll N\hbar$, the eigenstates localize in the momentum space and one obtains a Poisson distribution for the spectrum. The quantum dynamics has a time-reversal symmetry T for $\gamma=0$ (or $l\hbar$, l integer) and a parity symmetry P for $\phi=0$ (or $j\pi/N$, j integer). Though the T symmetry may be violated for $\gamma\neq 0$, still a more generalized antiunitary symmetry $S=TP=PT$ can be preserved in the system if $\phi=0$ [7,16]. By a slow variation of these parameters, one can realize the various intermediate stages of the statistical properties of quantum operators, for example, the Poisson spectrum can be obtained by K variation. One can also obtain four transitions from time reversal and parity symmetry: (i) P "fully" broken but T only "partly," (ii) P and T both fully broken but TP (an antiunitary symmetry like T) partly, (iii) T preserved but P partly broken, and (iv) TP preserved but T and P partly broken. In each case, a single parameter (γ or ϕ) can be identified to govern the partial symmetry breaking. The Brownian motion theory predicts COE$\to$CUE for (i) and (ii) and 2 COE$\to$COE for (iii) and (iv) under the random-matrix hypothesis. Assuming this hypothesis to be valid for strongly chaotic systems, in Ref. [16] we have obtained a semiclassical formulation for two-level spectral fluctuation measures for the transition cases as well as for the local parameters governing these transitions. We have also numerically verified the presence of these transitions in spectral fluctuations of QKR [16]. In this paper, we aim to do the same for strength fluctuations [only case (i) is dealt with here].

III. EFFECT OF SYMMETRY BREAKING ON EIGENVECTOR STATISTICS: A RANDOM-MATRIX APPROACH

The study of eigenvector statistics can be carried out by studying the probability distribution $P(y)$ of intensities or strengths $y_{ik}=N|\langle k|E_i\rangle|^2$ of eigenvector $|E_i\rangle$ in any arbitrary basis $|k\rangle$, $k=1,\ldots,N$. The numerical study of the distribution $P(y)$ (where subscript ik is dropped for simplicity) is facilitated by the fact that there are N^2 elements available for eigenvector statistics as compared to N elements for level statistics (provided by the diagonalization of an $N\times N$ Hamiltonian matrix, or, in our case, the time-evolution operator). Though the eigenvector statistics is inherently basis dependent, still it is believed [20] that for each system a class of "generic basis" exists, for which the eigenvectors have the same statistical properties.

For the universality classes, the orthogonal invariance of the ensemble can be used to obtain eigenvector components distribution. For Gaussian ensembles, the distribution $P(y)$ is given by [17]

$$P(y)=\frac{\Gamma(\beta N/2)}{N\Gamma(\beta/2)\Gamma(\beta(N-1)/2)}\left[\frac{y}{N}\right]^{(\beta/2)-1} \times\left[1-\frac{y}{N}\right]^{(\beta(N-1)/2)-1}, \tag{5}$$

where $\beta=1,2,4$ give the distribution for GOE, GUE, and GSE, respectively. In the semiclassical regime, $N\gg 1$, the above formula is well approximated by

$$P(y)=\frac{(\beta/2)^{\beta/2}}{\Gamma(\beta/2)}y^{(\beta/2)-1}e^{-\beta y/2} \quad (N\to\infty). \tag{6}$$

Here the strengths are asymptotically independent, distributed around a mean value $\langle y\rangle=N^{-1}$. Equation (6) is in fact the χ^2_β distribution with β degrees of freedom. The basic property of this distribution can be briefly mentioned as follows: if we consider β-independent random variables x_i, $i=1,\ldots,\beta$ and each variable has a Gaussian probability distribution with zero mean and variance $\sigma/\sqrt{\beta}$ then the sum of squares $y=\sum_{i=1}^{\beta} x_i^2$ will obey χ_β^2 distribution with mean value $\langle y \rangle = \sigma^2$. For $\beta=1$, this is known as Porter-Thomas distribution. The distribution $P(y)$, given by Eq. (6), is valid for circular ensembles too [17].

The universality classes are applicable to a system when a symmetry is exact or completely broken. It turns out that the ensemble theory for these universality classes is akin to that in equilibrium statistical mechanics where no attention is paid to the approach to equilibrium. This approach to equilibrium for partly broken symmetries is taken care of in Dyson's theory [14]. Here eigenvalues are treated as particles which perform a Brownian motion under a mutual two-body repulsive logarithmic potential as a parameter is varied. This result derives from, for example, parameter-dependent circular ensembles of unitary matrices $U_{\gamma+\delta\gamma}=e^{i\delta\gamma M/2}U_\gamma e^{i\delta\gamma M/2}$ (similarly for parameter-dependent Gaussian ensembles of Hermitian matrices $H_{\gamma+\delta\gamma}=H_\gamma+\delta\gamma M$) in which an infinitesimal random perturbation—in the present application a symmetry breaking—is applied at each time γ. Here γ can be thought of as "time"; in our problem, $\gamma=0$ is the symmetry-preserving case and $\gamma>0$ is the symmetry-breaking case. Stationarity is achieved in the $\gamma\to\infty$ limit when the appropriate universality class of the ensemble is attained. The Hermitian matrix M corresponding to the symmetry-breaking random perturbation belongs to a universality class of the Gaussian ensembles, which in turn fixes the $\gamma\to\infty$ limit [14–16].

Under the variation of a symmetry-breaking parameter, the eigenvectors $|E_i\rangle$'s are also affected. For example, the possibility of T-breaking detection in eigenvectors arises because the form of $|E_i\rangle$ changes from the time-reversal invariant (TRI) case (with only real components) to the time-reversal noninvariant case (with real and imaginary parts independently varying when γ, the "global" T-breaking parameters is varied). The essential transition in strength fluctuations is governed by the rate at which the eigenstates become complex as the strength of the TRNI part increases. As mentioned in Sec. I, in large-N) limits, this transition is abrupt with respect to parameter γ but one can define a "local" parameter Λ in terms of which it is smooth. The parameter Λ is basically the measure, in a region of the spectrum with average level spacing $D(E)$, of a small-γ contribution to the amplitude of the wave function from its imaginary part [15,16],

$$\Lambda=\frac{\gamma^2 v^2}{D^2}\ , \tag{7}$$

with v^2 as the variance of the matrix elements [of the symmetry-breaking part of $U(\gamma)$ in the diagonal representation of the symmetry-preserving part]. Note that γ depends only on the interaction H (or U) while Λ depends also on the level density in a complicated system. For given γ and v, the transition parameter is proportional to the matrix dimensionality, this arising from the effects of the distant levels which compress the spectrum, and thereby enhance by a factor $N^{1/2}$ the imaginary amplitude which is admixed into the real $\gamma=0$ eigenstate. It is because for N levels the spectrum span is proportional to $(\langle \mathrm{Tr}H^2\rangle/N)^{1/2}\propto N^{1/2}$ so that the spacing is proportional to $N^{-1/2}$. It implies that, at any point in the spectrum of a real system, there is defined a physically significant effective dimensionality which enhances the sensitivity of the fluctuation measures to the time-reversal breaking interaction. For any serious analysis, therefore, this dimensionality must be taken into consideration. As Eq. (7) indicates, the transition parameter Λ is also energy dependent, increasing, as one moves up in energy toward the spectrum centroid.

For quantum chaotic systems, in the semiclassical limit, Λ can also be expressed in terms of the actions of periodic orbits of underlying classical dynamics (see Ref. [16] for details) if the matrices associated with quantum motion are fully random, belonging to classical ensembles of matrices and the symmetry-breaking part of these matrices is also a random matrix in the diagonal representation of the symmetry-preserving part. In this paper, we show that the parameter Λ, governing the smooth transition in our numerical analysis, also turns out to be the same, i.e., it is in agreement with our semiclassical as well as RMT prediction.

For Gaussian-type Brownian ensembles, FKPT [11] gave an approximate formulation for the two-point strength fluctuation measures. They exploited the orthogonal invariance of the ensembles to find an appropriate form of the probability density $P(y)$ of one component of an eigenvector. They considered a GOE$\to$GUE transition represented by the ensemble of Gaussian random matrices of type $H(\alpha)=H(S)+i\gamma H(A)$, where S and A are N-dimensional real symmetric and antisymmetric Gaussian random matrices and are statistically independent of each other. Here, for $0\le\gamma\le 1$, $H(\gamma)$ defines an interpolating ensemble, and for $\gamma>1$ an extrapolating ensemble; $H(0)$ and $H(1)$ give the two limits of transition, namely, GOE and GUE, respectively. For all γ values, the ensembles are invariant under orthogonal transformations. The case $\gamma=\infty$ represents the ensemble of antisymmetric matrices, known as AGOE, and is also of some mathematical and physical interest.

To describe the FKPT method briefly, one proceeds by decomposing the jth eigenstate $|E_j\rangle$ in real ($|R_j\rangle$) and imaginary ($|I_j\rangle$) components,

$$|E_j\rangle=e^{i\delta_j}[t_j|R_j\rangle+i\sqrt{1-t_j^2}|I_j\rangle]\ . \tag{8}$$

Here t_j is the contribution of the real component and the phase δ_j is chosen so as to keep the real unit vectors $|R_j\rangle$ and $|I_j\rangle$ orthogonal. This gives the strength y, in an N-dimensional arbitrary basis k, as

$$y=N[t_j^2\langle k|R_j\rangle^2+(1-t_j^2)\langle k|I_j\rangle^2]\ . \tag{9}$$

Due to invariance of the Brownian motion ensembles under orthogonal transformations, the application of any such transformation on the eigenstates yields the eigenstates of another member of the ensemble. This ensures that the vectors $|R\rangle$ and $|I\rangle$ both uniformly cover the N-dimensional unit sphere. In the large-N limit, this enables one to express y as the sum of the squares of two zero-centered Gaussians with variances t_j^2 and $(1-t_j^2)$ with the following form of density [11]:

$$P(y|\theta_j)=\frac{1}{\sqrt{\theta_j}}\exp\left[-\frac{y}{\theta_j}\right]J_0\left[i\frac{\sqrt{(1-\theta_j)}}{\theta_j}y\right], \quad (10)$$

where θ_j $[=4t_j^2(1-t_j^2)]$ is the product of relative strengths of real and imaginary parts of the eigenvector component, $P(y|\theta_j)$ is the probability distribution of strengths for a fixed θ_j, and J_0 is Bessel's function. Now expressing $P(\theta_j)$ as the θ_j density, i.e., the probability of the occurrence of a given θ_j, the final strength density $P(y)$ can be obtained from $P(y|\theta_j)$ by averaging over θ_j (with $\bar{\theta}_j$ denoting the average of θ_j),

$$P(y)=\int_0^1 P(y|\theta_j)P(\theta_j)d\theta_j \quad (11a)$$

$$=\int_0^1\left[\sum_{n=0}^{\infty}\frac{(\theta_j-\bar{\theta}_j)^n}{n!}\frac{\partial^n}{\partial\bar{\theta}_j^n}P(y|\bar{\theta}_j)\right]P(\theta_j)d\theta_j \quad (11b)$$

$$=\left[1+\sum_{n=1}\frac{M_n}{n!}\frac{\partial^n}{\partial\bar{\theta}_j^n}\right]P(y|\bar{\theta}_j), \quad (11c)$$

where, in Eq. (11b), $P(y|\theta_j)$ is expanded in Taylor's series around $\bar{\theta}_j$ and Eq. (11c) is obtained by using the standard definition of moments M_n of θ_j distribution, that is, $M_n=\int_0^1(\theta_j-\bar{\theta}_j)^n P(\theta_j)d\theta_j$.

The first four moments of $P(y)$ distribution for large N can be given as follows:

$$\begin{aligned}&\bar{y}=1,\\&\text{var}(y)=2-\bar{\theta}_j,\\&K_3(y)=8-\bar{\theta}_j,\\&K_4(y)=48-48\bar{\theta}_j+6\bar{\theta}_j^2+9\,\text{var}(\bar{\theta}_j).\end{aligned} \quad (12)$$

The above formulas are useful only if they can be expressed in terms of Λ, the local symmetry-breaking parameter. For this purpose, French *et al.* used a perturbation theory for the variances (see Ref. [11] for details), which gives

$$\bar{\theta}_j=-\frac{2\pi^2\Lambda}{3}[\ln(2\pi^2\Lambda)+\gamma-2] \quad (13)$$

where γ is Euler's constant. These results are a good approximation for $\Lambda^{1/2}\le 0.2$.

Though the FKPT theory gave, in principle, a formulation for the distribution $P(y)$ of strengths, it requires prior information about θ_j distribution. The absence of an explicit formulation in terms of the local symmetry-breaking parameter Λ later on motivated many others to study this problem. Here we give briefly the results obtained by SI [22] as we will be using their results to compare with our numerical results for QKR. Their analysis consists in first obtaining the joint probability density of finding an eigenvalue E and the corresponding eigenvector $|E\rangle$, normalized to unity, by using orthogonal invariance of the ensembles. This is followed by an integration over all energies and all eigenvector components, except that one gives the required probability density $P(y)$ for one component of an eigenvector in terms of the parameter $\epsilon=2\gamma^2N=2\pi^2\Lambda$,

$$P(y;\epsilon)=\frac{\epsilon e^{\epsilon}}{\pi}\int_0^{\pi}\frac{\exp\{-[\epsilon+y(1-\cos\phi)]/\sin^2\phi\sin^2\theta\}}{\sin^4\theta\sin^3\phi}\times d\theta\, d\phi\,. \quad (14)$$

Here P is normalized such that both $\int_0^{\infty}P(y;\epsilon)dy=1$ and $\int_0^{\infty}yP(y;\epsilon)dy=1$. The parameter ϵ varies between 0 and ∞; $\epsilon\to 0$ gives the orthogonal limit and for $\epsilon\to\infty$ the unitary limit is obtained. The variance of y associated with the distribution $P(y;\epsilon)$ can be given as follows:

$$\text{var}(\epsilon)=2-\epsilon e^{\epsilon}E_1(\epsilon), \quad (15)$$

where E_1 is the exponential integral. It interpolates between 2 for the GOE and 1 for GUE. For $\epsilon\to 0$, it goes like $2+\epsilon\ln\epsilon$.

The strength density in the small-strength region is more sensitive to symmetry breaking than any other spectral or strength two-point measures. Briefly, it can be explained as follows. The COE strength density exhibits a $y^{-1/2}$ singularity at $y=0$ for all N. The expected number of small strengths is thus very different for the COE and CUE. For example, with large N, 11% of the COE but only 2% of the CUE strengths is expected to be less than 0.02. Thus it is of significance to know how rapidly the probability of small strengths varies with Λ. The best measure, for this purpose, is the distribution function for small strengths which can be defined as follows [11,18]:

$$F(x;\varepsilon)=\int_{\varepsilon}^{x}P(y)dy=\int_0^{x}P(y)dy-\int_0^{\varepsilon}P(y)dy=F(x)-F(\varepsilon), \quad (16)$$

where x,y,ε are the strengths measured in units of the mean strength and ε is a cutoff, which is needed in the analysis of experimental data and depends on the quality of the data. (The need to introduce ε arises due to measurement or numerical errors associated with the calculation of smallest strengths). The optimal x depends on the amount of data available and is sensitive to missing and spurious levels. Briefly, since there is a singularity in the strength density at $\Lambda=0$, the smaller the value x considered, the more rapidly the transition proceeds as a parameter Λ. However, experimental and sample error considerations put a limit on the smallest x considered as well as ε too. We have chosen $\varepsilon=0.004$ and $x=0.05$ to make use of theoretical (RMT) numbers given in Ref. [18].

IV. NUMERICAL ANALYSIS

The main assumption underlying the transition theory in RMT is that the Hermitian operator M in the symmetry-breaking part $\exp(i\gamma M)$ of the matrix U_γ is random when expressed in the diagonal representation of the symmetry-preserving part of U_γ [16]. In our earlier work [16], we have verified that for the choice of parameter values $K/4\pi^2\simeq 20\,000$ with $N\simeq 200$ and $\hbar=1$ in

QKR (strong-chaos limit), which ensures the delocalization of eigenfunctions and strongly chaotic nature of classical dynamics, the matrix M turns out to be random and belongs to the class of GUE ensembles and therefore transition theory is valid for this range of K. In general, for quantum chaotic systems, this theory is valid if the underlying classical dynamics is strongly chaotic and quantum dynamics is fully delocalized for exact symmetry cases and remains so while the symmetry-breaking parameter is slowly varied [16]. This suggests carrying out the numerical study, of the transition in eigenvector components of QKR, around this K value as the fingerprints of RMT-type transition behavior should be expected only in the range of validity of the random-matrix hypothesis. Furthermore, due to strong sensitivity of the quantum dynamics to the stochastic parameter K, a small variation of K in this range produces independent matrices. We use this property to numerically generate the matrix ensembles; the reason for using the ensemble of matrices instead of just one is to improve the statistics and minimize finite sample-size effects. For our numerical analysis, we take ensembles of three matrices of dimension $N=199$ with $K/4\pi^2=20\,002$, $20\,003$, and $20\,004$, $\hbar=1$, and $T=1$.

Here we confine ourselves to study only the effect of T breaking on eigenvector components and therefore set the parity symmetry parameter $\phi=\pi/2N$ (which fully breaks the P symmetry, as mentioned in Sec. II.). Now, in order to see a smooth transition, it is necessary to identify the proper T symmetry-breaking parameter Λ [Eq. (7)] for our system. Moreover, one should also know the relation between γ, the global symmetry-breaking parameter (the one which is present in U), vs Λ, the local symmetry-breaking parameter, as well as the dependence of γ on N. The latter is required due to strong sensitivity of transition to size of the sample; a small change in sample size may bring an abrupt transition for the same γ values. For our case, as Eq. (3) indicates, the T-breaking part of U is simply the operator $\exp(i\gamma p)$ and the associated Hermitian operator $M=p$. Here p, when expressed in the diagonal representation of the T symmetry-preserving part of U, turns out to be an antisymmetric matrix. Moreover, for strongly chaotic cases of QKR, the matrix p is a random matrix (see Ref. [16] for details) with distinct matrix elements distributed as independent zero-centered Gaussian variables with zero mean and variance $\mathrm{Tr}[p^2/N(N-1)]\simeq N/12$. Hence from Eq. (7), the Λ for T breaking in QKR [16] is given by (with $D=2\pi/N$ for circular ensembles)

$$\Lambda=\frac{\gamma^2N^2}{4\pi^2}\,\mathrm{Tr}\left[\frac{p^2}{N(N-1)}\right]=\frac{\gamma^2N^3}{48\pi^2}\ . \tag{17}$$

The study of spectral fluctuations for the COE$\to$CUE transition in RMT tells us that the equilibrium is achieved (i.e., the CUE limit) for $\Lambda\simeq0.2$ [11,16]. Expecting to see the transition here also nearly in the same Λ range, we slowly break the T symmetry by varying γ in the range 0.0–0.004 (step size $\delta\gamma\simeq0.0004$). For each γ chosen in this range, we generate an ensemble of three matrices, the diagonalization of which gives us $3N$ independent sets of eigenfunctions. To see the effect of the γ variation on $P(y)$, all eigenvectors are considered together (thus giving $3\times199\times199$ components for statistics) at each γ; the resulting histograms for $P(y)$ are plotted in Fig. 1 for some of these cases. For a clear view of the transition, we also plot the two limiting $P(y)$ distributions for RMT, namely, COE and CUE [given by Eq. (6)]. Figures 1(a)–1(d) show that, while for $\gamma=0$ the histogram does coincide with the theoretical COE curve, it lies between COE and CUE curves for $\gamma\neq0$. With increasing γ, this histogram slowly approaches the theoretical CUE limit. For γ very large (e.g., $\gamma\simeq1$), it finally coincides with the CUE curve. For comparison with random-matrix results for transition ensembles, the $P(y)$ vs y relation, given by Eq. (14), is also fitted to each of these histograms; hence we have used the relation $\epsilon=2\pi^2\Lambda$. As shown in Fig. 1, we find that, for this choice of ϵ vs Λ relation, the RMT curve coincides well with the histogram for all y's. Thus the RMT formulation, given by Eq. (14) for $P(y)$ distribution in symmetry-breaking cases, seems to be a good model for the corresponding cases in quantum chaotic systems. Furthermore, the smooth variation of histogram between the COE$\to$CUE limit indicates that the probability of finding a given small strength ($y\leq0.02$) decreases with Λ but it increases for large strengths (e.g., $y\simeq1$), which is similar to the case in RMT. This implies that there is a larger tendency to have higher strengths with increasing Λ. This can be understood as a result of eigenvectors getting more and more complex with increasing Λ. The rate at which this probability changes with y decreases with Λ. Moreover, the results shown in Fig. 1 reconfirm that Λ defined by Eq. (7) is indeed the correct parameter to study the smooth transition in the kicked rotor; this was also confirmed by our earlier studies of the effect of symmetry breaking on eigenvalues of QKR [16].

To reconfirm that the transition in the distribution $P(y)$ has a similar behavior as for RMT, we also calculate var(y) measure for eigenvectors. Figures 2(a) and 2(b) show the dependence of var(y), for QKR, on small values of Λ and large values of Λ, respectively, for two matrix dimensionalities, namely, $N=199$ and 99 along with the corresponding RMT result given by Eq. (15). Figure 2(a) also contains the var(y) result [Eq. (12)], corresponding to the RMT theory given by French *et al.* [11] (the dashed curve); here the curve beyond $\Lambda^{1/2}\geq0.2$ is obtained by extrapolation. The solid curve corresponds to the results obtained by Sommers and Iida [Eq. (15)] [22]; here again we have used the relation $\epsilon=2\pi^2\Lambda$. As seen in Fig. 2(a), the numbers for var(y) are in better agreement with the perturbation results [Eq. (12)] [11] for $\Lambda^{1/2}\leq0.2$ as compared to the one given by Eq. (15). Figure 2(b), showing the large-Λ behavior of var(y), seems to indicate a near agreement between our numerical results for QKR and those given by Eq. (15), for the above-mentioned relation between Λ and ϵ. To check the validity of FKPT and SI theories, we apply the χ^2 test. The calculated χ^2 values for FKPT and SI theories, for $N=199$ and 5 degrees of freedom (i.e., 5 data points), turn out to be 0.004 and 0.02, respectively, which are much less than the theoretical χ^2 ($=11.07$) at 5% level of significance. This implies an excellent agreement of both the theories with the results obtained for QKR; a relatively lower χ^2 value for FKPT as compared to SI theory indicates a better correspondence. To compare the SI curve with QKR data, given in Fig. 2(b), the χ^2 test is applied again; the χ^2 value for 9 degrees of freedom turns out to be 0.03, which is much less than that of theoretical χ^2 $(=16.9)$ at 5% level of significance, confirming once again the validity of SI theory. Furthermore this figure also indicates that here the approach to equilibrium, i.e., the CUE limit $[\text{var}(y)=1]$ is very slow; for example, for $\delta\,\text{var}(y)/\delta\Lambda \simeq -5$, $\Lambda \simeq 0.03$, while at $\Lambda \simeq 0.15$, $\delta\,\text{var}(y)/\delta\Lambda \simeq -0.8$. Note that the curves in Fig. 2(b) have the proper COE limit, namely, $\text{var}(y) \simeq 2$ for COE while approaching the CUE limit for $\Lambda \rightarrow \infty$.

For a better understanding of the effect of T symmetry breaking on small strengths, we study the $F(x;\varepsilon)$ measure too. The results are given in Fig. 3, which shows that the probability of occurrence of small strengths decreases very rapidly with increase in Λ, a behavior similar to that found in RMT. Thus $F(x;\varepsilon)$ can be a good measure for detecting small-symmetry violation in quantum chaotic systems too. For comparison of the $F(x;\varepsilon)$ measure with random-matrix results, we use Monte Carlo numbers given in Ref. [18] as theory; agreement is again good.

In Ref. [16], we verified that the transition in spectral fluctuations takes place in accordance with the relation $\gamma \propto N^{-3/2}$ with γ as a global parameter and N dimension. To verify that this formulation is valid for strength fluctuations also, the above study is repeated for dimension $N=99$ too. The results obtained are also plotted in Figs. 2(a), 2(b), and 3, which show that the points for $N=99$, for both $\text{var}(y)$ and $F(x)$, are also in good agreement over the entire range of Λ; the application of the χ^2 test also confirms our observation. This implies that the transition in strength fluctuations is indeed governed by the same parametrization. But note that the transition in strength fluctuations completes at a much higher value of Λ. Here equilibrium is achieved for $\Lambda \gtrsim 0.3$ while for spectral fluctuations one attains the CUE limit for $\Lambda \simeq 0.2$ [11,16]. Thus the speed of transition seems to be different for strength and spectral fluctuations, which is contrary to the behavior observed in RMT.

Although the RMT results for eigenvector statistics are still not fully available and it is not possible to directly confirm the existence of transition ensembles of RMT in the eigenvector statistics of quantum chaotic systems, the coincidence of the limits and transition cases for the probability of one component of the eigenvector as well as the presence of a similar transition parameter Λ as given by RMT indicates the existence of Brownian en-

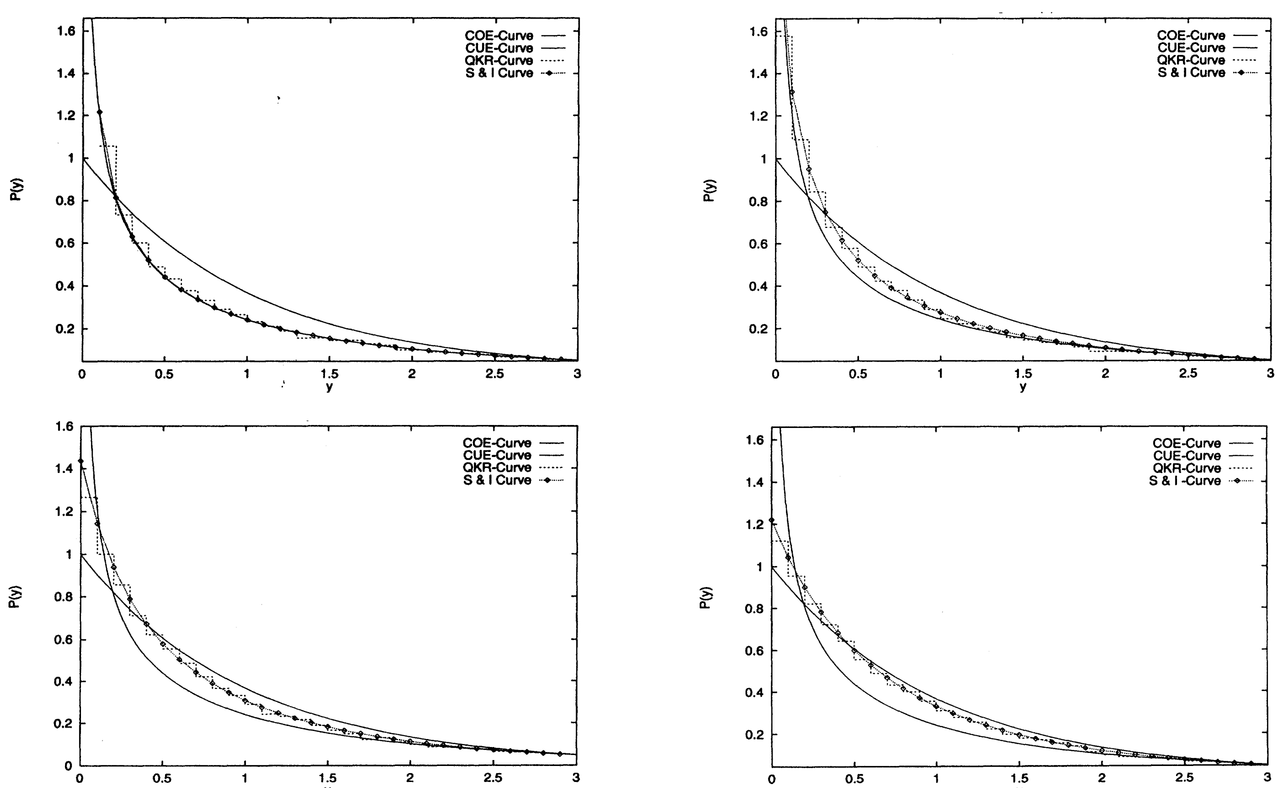

FIG. 1. The histogram (dashed line) for the distribution of eigenvector components $P(y)$ vs y for T violation. The data correspond to the eigenvector components of three matrices for K around 20000, $N=199$, $\hbar=1$, $\phi=\pi/2N$, and γ varying; (a) $\gamma=0.0$, (b) $\gamma=0.0008$, (c) $\gamma=0.0016$, (d) $\gamma=0.0024$. The two solid lines are two limits (the one with larger convexity is for COE and the other for CUE) of COE$\rightarrow$CUE transition. The dashed line is the fitting given by Eq. (14).

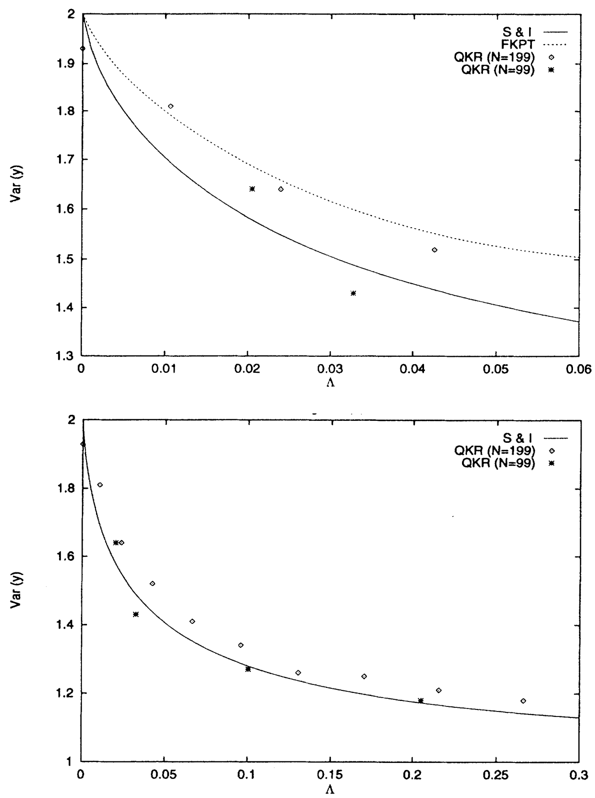

FIG. 2. The variance var(y) vs Λ for T violation. (a) With respect to small Λ. Here the points are given for two values for N, namely, $N=199$ and 99, along with statistical error bars which are obtained by taking three times of standard errors of variance (3σ error) associated with each data point where $\sigma \simeq 0.01$ and 0.02 for $N=199$ and 99, respectively. The dashed curve is the perturbation RMT result, obtained by French *et al.* [Eq. (12)] while the solid curve is given by Eq. (15). (b) With respect to large Λ. The corresponding RMT curve is the one given by Eq. (15).

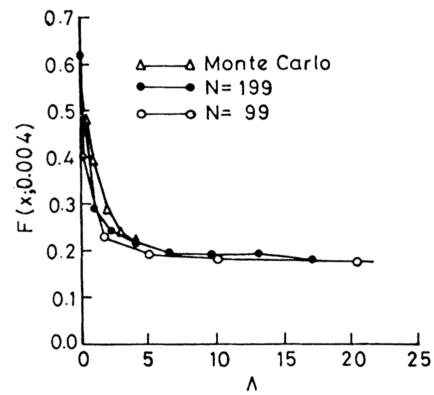

FIG. 3. The measure $F(x;0.004)$ vs Λ for T violation; here $x=0.05$. The points here correspond to two values of N, namely $N=199$ and 99. Also shown are the Monte Carlo points which agree well with the $N=199$ and 99 curve.

sembles here as well. It can similarly be verified for other symmetry breakings. Furthermore, in a separate study of the distribution of zeros of chaotic eigenfunctions, expressed in phase-space representation [25], we have again observed the manifestation of random-matrix-type behavior. We have also verified the presence of Brownian ensembles of RMT in spectral correlations of quantum chaotic systems [16]. Thus it is becoming more and more evident that, in the semiclassical limit, the random-matrix-type behavior seems to be present in almost all the statistical properties of quantum operators associated with a classically chaotic dynamics [25]. Intuitively it may be interpreted as the randomness, introduced in quantum properties, by the chaotic nature of underlying classical dynamics where the degree and the type of randomness is governed by the symmetries of the motion.

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